Boolean Combinations of Weighted Voting Games

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ABSTRACT

Weighted voting games are a natural and practically important class of simple coalitional games, in which each agent is assigned a numeric weight, and a coalition is deemed to be winning if the sum of weights of agents in that coalition meets some stated threshold. We study a natural generalisation of weighted voting games called *Boolean Weighted Voting Games* (BWVGs). BWVGs are intended to model decision-making processes in which components of an overall decision are delegated to committees, with each committee being an individual weighted voting game. We consider the perspective of an individual who has some overall goal that they desire to achieve, represented as a propositional logic formula over the decisions controlled by the various committees. We begin by formulating the framework of BWVGs, and show that BWVGs can provide a succinct representation scheme for simple coalitional games, compared to other representations based on weighted voting games. We then consider the computational complexity of problems such as determining the power of a particular player with respect to some goal, and investigate how the power of a player with respect to the overall goal is related to the power of that player in individual games. We show trade-offs between the complexity of these problems, the nature of underlying Boolean formulas used, and representations of weights (binary versus unary) in our games.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; I.2.4 [**Knowledge representation formalisms and methods**]

General Terms

Theory

Keywords

weighted voting games, games, complexity

1. INTRODUCTION

Recent years have seen an explosion of interest in computational issues surrounding coalitional games (see, e.g., [3, 1, 4, 5]) and voting systems (see, e.g., [6, 2]). *Weighted voting games* are an important class of systems at the intersection of voting and coalitional games [18]. In a weighted voting game, each voter is assigned a numeric weight, and a coalition is said to win if the sum of their weights meets or exceeds a certain stated threshold. Weighted voting games are widely used in many real-world decision-making bodies, and have a simple and elegant mathematical formulation. Many computational questions surrounding weighted voting games have been considered [3, 4].

In this paper, we are concerned with the following natural generalisation of weighted voting games. We consider decision making via multiple committees, where each committee has the authority to decide the outcome (either "yes" or "no") to a single issue. Each committee is a weighted voting game, and while the same individuals may appear in multiple committees, their weights may be different in different committees; different committees may also have different threshold values. Now, consider someone who has an interest in the decisions that are being made by the component committees. Such an individual will typically have a *goal* that she desires to be satisfied, and moreover this goal will often have some logical structure. For example, a typical goal with respect to political decision making might be *(raise university funding <u>or</u> raise healthcare funding) <u>and</u> <u>not</u> raise taxes*. Many natural questions arise in such a setting. For example: Which coalitions might be able to bring the goal about? How important is a particular individual with respect to the achievement of the goal? Can we derive the answers regarding the global goal from those for the subgoals?

The same model has applications in the context of multiagent systems: Consider, for example, a group of agents, each with some amounts of resources of several types, and a task that requires some resources to be completed. Since there may be many ways of completing the task, its resource requirements are best described via a propositional formula such as *(at least T_1 units of R_1) <u>or</u> ((at least T_2 units of R_2) <u>and</u> (at least T_3 units of R_3))*, where the propositions are weighted voting games describing which coalitions have enough resources of each given type.

In short, the aims of the present paper are, first, to formulate *Boolean Weighted Voting Games*, and second, to investigate their computational properties. We show that while Boolean weighted voting games are more expressive than, say, vector weighted voting games, in many settings (particularly when the underlying goal formulas are monotone) this gain in expressiveness comes without a price in terms of computational complexity (as compared to vector weighted voting games). Naturally, unrestricted Boolean weighted voting games can lead to increased complexity. For several natural problems, we show trade-offs between computational complexity, the nature of underlying Boolean formulas used in BWVGs, and representations of weights (binary versus unary).

2. PRELIMINARY DEFINITIONS

Propositional Logic. Let $\Phi = \{p, q, \ldots\}$ be a (fixed, nonempty)

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vocabulary of Boolean variables, and let $\mathcal{L}$ denote the set of (well-formed) formulas of propositional logic over Φ, constructed using the conventional Boolean operators ("$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$", and "$\neg$"), as well as the truth constants "$\top$" (for truth) and "$\perp$" (for falsity). If "$\wedge$" and "$\vee$" are the only operators appearing in formula φ then we say that φ is *monotone*. We assume a conventional semantic consequence relation "$\models$" for propositional logic. A subset ξ of Φ is a *valuation*, and we write $\xi \models \varphi$ to mean that φ is true under, or satisfied by valuation ξ. (One may interpret a valuation ξ for a formula φ as setting all φ's variables from ξ to $\top$ and all the remaining ones to $\perp$.) The *size* of a Boolean formula φ (denoted by $|\varphi|$) is the number of variable and constant occurrences in φ. For example, the size of the formula $x \vee (y \wedge \neg x)$ is 3. Note that under any reasonable representation formalism one can represent a formula of size s using $O(s \log |\Phi|)$ bits.

Coalitional Games. A *coalitional game* $G = (N, v)$ is described by a set of *players* N and a *characteristic function* $v : 2^N \rightarrow \mathbb{R}$ that for each subset, or a *coalition*, of players $C \subseteq N$ outputs the value associated with this coalition. Intuitively, $v(C)$ is the value that the members of C can achieve by working together [15]. A coalitional game is called *simple* if $v(C) \in \{0, 1\}$ for all $C \subseteq N$. In such games, we say that C *wins* if $v(C) = 1$ and C *loses* otherwise. A simple game G is *monotone* if $v(C) = 1$ implies $v(C') = 1$ for any $C' \supseteq C$. (Some authors use the term "simple game" to refer to monotone games only; in this paper we consider both monotone and nonmonotone simple games and explicitly discuss how restricting attention to monotone games would affect our results.) In the context of monotone games, a useful notion is that of a *maximal losing coalition*: C is said to be a maximal losing coalition if $v(C) = 0$ but $v(C') = 1$ for any $C' \supset C$. Clearly, a monotone game can be completely described by listing its maximal losing coalitions. We consider games with finite numbers of players only: we assume $|N| = n$ and write $N = \{1, \ldots, n\}$.

One can represent a coalitional game $G = (N, v)$ by listing the values $v(C)$ for all $C \subseteq N$. However, the size of this representation will be exponential in the number of players n. It is therefore important to identify and study compact representations of practically important coalitional games. One such class of games is that of *weighted voting games* (WVGs)—see, e.g., [18]. A weighted voting game is given by a set $N = \{1, \ldots, n\}$ of *players*, a list of n *weights* $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, and a *threshold* $T \in \mathbb{R}$. When the set of players N is clear from the context, we use the notation $g = (T; w_1, \ldots, w_n)$ to denote a WVG g with the set of players N, threshold T, and weights $\mathbf{w}$. We write $w(C)$ to denote the total weight of coalition C under the weight vector $\mathbf{w}$, i.e., $w(C) = \sum_{i \in C} w_i$. The characteristic function v of a game $(T; \mathbf{w})$ is given by $v(C) = 1$ if $w(C) \geq T$ and $v(C) = 0$ otherwise. Thus, any WVG is a simple game. Note that if all weights are nonnegative, the game is monotone (but the converse is not necessarily true).

A more general class of simple games, also studied in the literature, is that of *k-vector weighted voting games (k-VWVG)* [17]. Intuitively, these games are intersections of k distinct weighted voting games: to win in such game, a coalition has to win in all of the underlying games. More formally, a k-vector weighted voting game is given by a set of players N, a list of k weight vectors $\mathbf{w}^1, \ldots, \mathbf{w}^k$, where $\mathbf{w}^i = (w_1^i, \ldots, w_n^i)$ for each $i \in \{1, \ldots, k\}$, and a list of k thresholds $T^1, \ldots, T^k$. A coalition $C \subseteq N$ wins ($v(C) = 1$) if $w^i(C) \geq T^i$ for each $i \in \{1, \ldots, k\}$ and loses ($v(C) = 0$) otherwise. It is well-known that any simple game G can be represented as a k-VWVG for a sufficiently large k and an appropriate set of weights, but k may need to be exponential in n. Given a simple game G, the smallest k such that G can be represented as a k-VWVG is called the *dimension* of G, and is denoted by $\dim(G)$.

Computational Complexity. We assume that the reader is familiar with basic notions of computational complexity, such as classes NP and coNP, levels of the polynomial hierarchy such as Σ_2^p and Π_2^p, nondeterministic polynomial-time Turing machines (NP-machines), and many-one polynomial-time reductions (see, e.g., [16]). In addition to these well-known concepts, we also make use of some less well-known ones, as follows. A language L is in D^p if $L = L_1 \cap L_2$, for some languages $L_1 \in$ NP and $L_2 \in$ coNP. Similarly, a language L is in D_2^p if $L = L_1 \cap L_2$ for some languages $L_1 \in \Sigma_2^p$ and $L_2 \in \Pi_2^p$. We say that a function f belongs to #P if there exists an NP-machine M such that for each input x it holds that M on x has exactly $f(x)$ accepting computation paths. A language L belongs to class UP if its characteristic function is in #P, i.e., if there exists an NP-machine that on each member of L has exactly one accepting path and that on each nonmember of L has no accepting paths. Clearly, UP $\subseteq$ NP and it is believed that the subset relation is strict.

3. BOOLEAN WEIGHTED VOTING GAMES

The basic idea behind *Boolean weighted voting games* (BWVGs) is that we have a *collection* of weighted voting games $\mathcal{G} = \{g^1, \ldots, g^m\}$, with each game g^i over the same set of players N. (Note that players' weights and thresholds may differ between g^is.) Each individual weighted voting game g^i decides on a particular issue, or proposition, p^i. Informally, think of each weighted voting game g^i as being a committee that decides on whether a particular proposal p^i is implemented ($p^i = \top$) or not ($p^i = \perp$). That is, a coalition C that wins in g^i is able to choose the value for p^i. We let $\Phi = \{p^1, \ldots, p^m\}$ be the set of propositions corresponding to weighted voting games. A coalition C *controls* a variable p^i if it wins the corresponding weighted voting game g^i. Let Φ_C denote the set of propositions controlled by C, i.e., $\Phi_C = \{p^i \mid C \text{ wins in } g^i\}$. A *goal* is a propositional formula φ over the variables Φ corresponding to games $\mathcal{G}$.

Collecting these components together, a *Boolean weighted voting game* (BWVG) is a tuple $G = \langle N, \mathcal{G}, \Phi, \varphi \rangle$, where:

1. $N = \{1, \ldots, n\}$ is a set of players;
2. $\mathcal{G} = \{g^1, \ldots, g^m\}$ is a set of weighted voting games over N, where the jth game, g^j, is given by a vector of weights $\mathbf{w}^j = (w_1^j, \ldots, w_n^j)$ and a threshold T^j; we will refer to the games in $\mathcal{G}$ as the *component games* of G;
3. $\Phi = \{p^1, \ldots, p^m\}$ is a set of propositional variables, with each variable p^j corresponding to the weighted voting game g^j; and
4. φ is a propositional formula over Φ.

We will sometimes abuse notation and use the identifiers of the component games instead of variables in φ. For example, we may write $g^1 \wedge g^2$ instead of $\langle N, \{g^1, g^2\}, \{p^1, p^2\}, p^1 \wedge p^2 \rangle$.

When does a coalition win in a Boolean weighted voting game? Informally, we think of a coalition as winning if it is able to fix the variables under its control in such a way that the goal formula φ is guaranteed to be true. More formally, we say that C wins in G if

$$\exists \xi_1 \subseteq \Phi_C : \forall \xi_2 \subseteq (\Phi \setminus \Phi_C) : \xi_1 \cup \xi_2 \models \varphi.$$

The definition of Boolean weighted voting games requires some discussion. First, note that, following the standard definition of weighted voting games [18], we allow component games of

BWVGs to contain negative weights. In consequence, BWVGs constitute quite a powerful formalism to express nonmonotone games (see, e.g., Theorem 4). Yet, in many settings, it is natural to restrict attention to monotone BWVGs whose component games contain nonnegative weights only. One might then wonder if some of the complexity of BWVGs does not stem from the expressive power of negative weights. However, all our hardness results for BWVGs are proved using BWVGs with nonnegative weights only. On the other hand, our positive results often apply to games with negative weights. Throughout this paper we always clearly indicate settings where negative weights are relevant.

It is also interesting to consider BWVGs with respect to the logical structure of the goal formula. For example, it is easy to see that k-vector weighted voting games are simply BWVGs in which the goal formula is a conjunction of component game variables. In the next section we will show that general BWVGs can represent natural simple games in a considerably more compact fashion that VWVGs, even when restricted to monotone formulas. In further sections we will explore computational-complexity trade-offs related to the complexity of underlying formulas, as well as to representations of the weights of component games. In effect, we will see that, at least when limited to, e.g., monotone formulas, BWVGs offer a more expressive formalism (compared to VWVGs) without necessarily paying the price of higher computational complexity.

Representational Complexity. We mentioned above that any simple game with n players can be represented as a k-vector weighted voting game for $k = O(2^n)$, and hence as a BWVG with $O(2^n)$ component games. We will now present a simple counting argument showing that we cannot improve this worst-case behavior via using BWVGs. However, later in this section we show that for certain natural games BWVGs do offer a more compact representation than VWVGs (sometimes exponentially more compact).

PROPOSITION 1. *The total number of Boolean weighted voting games with $|N| = n$ and $|\varphi| = s$ is at most $2^{O(sn^2 \log(sn))}$.*

PROOF. Any individual weighted voting game can be represented using integer weights whose absolute values do not exceed $2^{O(n \log n)}$ [14]. We can assume without loss of generality that $|\mathcal{G}| = |\Phi|$ and $|\Phi| \leq |\varphi| = s$. Hence, given a BWVG G with n players and $|\varphi| = s$, we can find an equivalent representation of the same game that uses at most $O(sn^2 \log n)$ bits to represent all weights in all component games of G, and another $O(s \log s)$ bits to represent $\mathcal{G}$, Φ, and φ. Thus the total number of distinct games that can be represented as BWVGs with $|N| = n$ and $|\varphi| = s$ is $2^{O(sn^2 \log(sn))}$. □

COROLLARY 2. *For large enough n, there exist simple games with n players that cannot be represented by BWVGs with $|\varphi| < 2^n/n^5$.*

PROOF. There is a one-to-one correspondence between simple games with n players and binary vectors of length 2^n. Hence, there are exactly 2^{2^n} simple games. On the other hand, by Proposition 1, there are at most $2^{2^n/n}$ distinct BWVGs with $|\varphi| < 2^n/n^5$. □

However, there are natural simple games capturing realistic voting scenarios that can be represented much more compactly with BWVGs instead of VWVGs. Our first example considers a game with n players that can be represented as a disjunction of just two WVGs, but needs $\Omega(n)$ component games to be represented as a VWVG.

THEOREM 3. *Consider a BWVG $G = \langle N, \mathcal{G}, \Phi, \varphi \rangle$, where $\mathcal{G} = \{g^1, g^2\}$, $g^1 = (k; 1, 0, \ldots, 1, 0)$, $g^2 = (k; 0, 1, \ldots, 0, 1)$, $|N| = 2k$, and $\varphi = p^1 \vee p^2$. To represent G as a conjunction of m weighted voting games requires $m \geq k/2$ component games.*

PROOF. Observe that for a coalition C to win in G, it has to contain either all even-numbered players or all odd-numbered players. Hence, any maximal losing coalition (MLC) in G is of the form $(N \setminus \{2i, 2j-1\})$, where $i, j \in \{1, \ldots, k\}$. Denote such a coalition by $C_{i,j}$. Observe that there are exactly k^2 MLCs. We say that two MLCs $C_{i,j}$ and $C_{i',j'}$ *clash* if $i = i'$ or $j = j'$, i.e., if $C_{i,j} \cup C_{i',j'} \neq N$. Now, suppose that G can be represented as $\langle N, \{h^1, \ldots, h^m\}, \{q^1, \ldots, q^m\}, q^1 \wedge \ldots \wedge q^m \rangle$ with $m < k/2$. Each MLC has to lose in at least one of the component games $h^1, \ldots, h^m$. Hence, by the pigeonhole principle, there must be at least one component game (without loss of generality, h^1) that is lost by at least $2k$ distinct MLCs. Now, fix an arbitrary MLC $C_{i,j}$ that loses in h^1. Among the $2k$ MLCs that lose in h^1, there can be at most $k-1$ other MLCs of the form $C_{i,j'}$, $j' \neq j$, and at most $k-1$ other MLCs of the form $C_{i',j}$, $i' \neq i$. Hence, there must be an MLC $C_{x,y}$ that loses in h^1 and does not clash with $C_{i,j}$. Let $h^1 = (T; w_1, \ldots, w_n)$. We have

$$w(N) - w_{2i} - w_{2j-1} < T; w(N) - w_{2x} - w_{2y-1} < T. \quad (1)$$

On the other hand, both $C_{i,j} \setminus \{2y-1\} \cup \{2i\}$ and $C_{x,y} \setminus \{2i\} \cup \{2y-1\}$ are winning coalitions in G and hence in h^1 (the former contains all even-numbered players and the latter contains all odd-numbered players). Hence we have

$$w(N) - w_{2j-1} - w_{2y-1} \geq T; w(N) - w_{2i} - w_{2x} \geq T. \quad (2)$$

Equations (1) and (2) give a contradiction, and so $m \geq k/2$. □

The game considered in Theorem 3 shows that BWVGs can be considerably more succinct than VWVGs. However, the savings achieved by moving from VWVGs to BWVGs in this case are only polynomial: indeed, this game has $n^2/4$ MLCs, so by a simple argument, presented, e.g., in [18], we can represent it as an $n^2/4$-vector weighted voting game. Our next example illustrates that sometimes BWVGs can provide a representation that is exponentially more succinct than VWVGs. The game considered in the next theorem is nonmonotone, and we thus use negative weights in its BWVG representation.

THEOREM 4. *For any $n_0 > 0$, there is an $n \geq n_0$ and a game $G = \langle N, \mathcal{G}, \Phi, \varphi \rangle$ with $|N| = n$, such that $|\mathcal{G}| = O(n)$, but $\dim(G) = \Omega(2^{n-1})$.*

PROOF. Fix $n_0 > 0$, and set $n = n_0$ if n_0 is even and $n = n_0 + 1$ if n is odd. Let G be a game with n players in which a coalition C is winning iff its size $|C|$ is odd. The dimension of this game is $\Omega(2^{n-1})$ [18]. To obtain a compact representation of G as a Boolean weighted voting game, set $\Phi = \{p^1, p^2, \ldots, p^{n-1}, p^n\}$, $\varphi = (p^1 \wedge p^2) \vee \cdots \vee (p^{n-1} \wedge p^n)$, and $\mathcal{G} = \{g^1, \ldots, g^n\}$, where for all $k = 1, \ldots, n/2$ we have $g^{2k-1} = (2k-1; 1, \ldots, 1)$, $g^{2k} = (-(2k-1); -1, \ldots, -1)$.

Indeed, each subformula of the form $p^{2k-1} \wedge p^{2k}$ expresses that a coalition of size $2k-1$ wins in G; by taking the disjunction of all such expressions, we accommodate all winning coalitions. □

Basic Decision Problems in BWVGs. An important criterion in evaluating a representation formalism for coalitional games is the difficulty of answering natural questions relating to the entire game or specific players in this game, such as, e.g., deciding whether a given coalition is winning, or deciding whether a particular player is a dummy (see definitions below). To formally address the computational complexity of these decision problems in this context, we assume that all players' weights in each of the component games are integers given in binary, so a polynomial-time algorithm is one whose running time is polynomial in the number of players n, the size of the formula s and $\log W$, where $W = \max\{|w_i^j| \mid i = 1, \dots, n, j = 1, \dots, m\}$. However, we will also study the complexity of our problems in the important special case where the weights can be assumed to be small. In this setting, we will be interested in *pseudopolynomial-time* algorithms, i.e., algorithms whose running time is polynomial in n, s, and W; observe that such algorithms run in time polynomial in the size of the input if the weights are given in unary, or it is known that W is polynomially bounded in n and s. We will also discuss the complexity of our problems in settings where one of the natural parameters of the BWVG in question, e.g., the number of component games involved, can be assumed to be constant, or the formula φ is monotone.

Winning Coalitions. Given a BWVG, one of the most important questions to ask is whether a given coalition is winning. Given a coalition C, it is easy to decide which variables it controls and thus the real difficulty lies in choosing the values for those variables. If the underlying formula is of a particularly convenient form (e.g., is monotone or the number of its variables is bounded by a constant) then testing whether a coalition is winning is easy (for monotone formulas we can assume that all variables controlled by C are set to $\top$ while all other variables are set to $\perp$ and for formulas with few variables we can afford to enumerate all possible truth assignments). On the other hand, for the case of unrestricted formulas readers familiar with the theory of computational complexity will immediately see an easy reduction from the decision problem $\text{QSAT}_{2,\exists}$ [16, p. 428] directly implying the following result.

THEOREM 5. *Given a game $G = \langle N, \mathcal{G}, \Phi, \varphi\rangle$ and coalition $C \subseteq N$, deciding whether C wins in G is Σ_2^p-complete. This result holds even if there are only 2 players and the weights of all players in all component games are in $\{0, 1\}$. However, the problem is in P if the underlying formula is monotone.*

Swing Players. Our next question concerns pairs of the form (C, i), where $C \subseteq N$ and $i \in N \setminus C$. A standard notion in simple games is that of a *swing* player: i is a swing player for C in game G if C loses in G but $C \cup \{i\}$ wins in G. In the SWING-PLAYER problem we are given a game $G = \langle N, \mathcal{G}, \Phi, \varphi\rangle$, a coalition $C \subseteq N$, and a player $i \in N$, and we ask whether i is a swing player for C in G. Again, we can easily compute which variables are controlled by C and $C \cup \{i\}$, so the problem is easy if φ is monotone or its size is bounded a constant. However, in general the problem is computationally hard.

THEOREM 6. SWINGPLAYER *is D_2^p-complete. The hardness result holds even if there are only 3 players and the weights of all players in all component games are in $\{0, 1\}$. However, the problem is in P if the underlying formula is monotone.*

PROOF. The case of monotone formulas follows directly from Theorem 5. Let us now consider the general case. To establish membership in D_2^p, we must exhibit two languages, L_1 and L_2, such that: (*i*) $L_1 \in \Sigma_2^p$; (*ii*) $L_2 \in \Pi_2^p$; and (iii) SWINGPLAYER $= L_1 \cap L_2$. For membership in D_2^p, define:

$$L_1 = \{\langle G, C, i\rangle : C \cup \{i\} \text{ wins in } G\}$$
$$L_2 = \{\langle G, C, i\rangle : C \text{ does not win in } G\},$$

where in each $\langle G, C, i\rangle$, G is a BWVG, C is a coalition of players from G, and i is a player in G. Clearly, $L_1 \in \Sigma_2^p$ and $L_2 \in \Pi_2^p$. By definition, $L_1 \cap L_2$ is the language of SWING PLAYER. To show D_2^p-hardness, we provide a reduction from the D_2^p-complete problem SAT_2^Σ-UNSAT_2^Σ, an obvious generalisation of the D^p-complete SAT-UNSAT problem [16, p. 415]. Without loss of generality, we can assume that an instance I of this problem is given by a pair of quantified Boolean formulas

$$I = \langle \exists \bar{u} \forall \bar{v} : \chi(\bar{u}, \bar{v}), \forall \bar{x} \exists \bar{y} : \neg\psi(\bar{x}, \bar{y})\rangle,$$

in which the variable sets ($\bar{x}$, $\bar{y}$, $\bar{u}$, and $\bar{v}$) are mutually disjoint. I is a "yes"-instance if both formulas are true and a "no"-instance otherwise. Given an instance I as above, we set $N = \{1, 2, 3\}$, create $|\bar{u} \cup \bar{v} \cup \bar{x} \cup \bar{y}|$ games corresponding to the variables in the problem instance, and one additional game corresponding to a variable w. We fix the games so that the coalition $\{1\}$ controls variables $\bar{x}$ and the coalition $\{1, 2\}$ controls $\bar{x} \cup \bar{u} \cup \{w\}$ (e.g., we can set $g^{x_k} = (1; 1, 0, 0)$ for all $x_k \in \bar{x}$, $g^{u_k} = g^w = (2; 1, 1, 0)$ for all $u_k \in \bar{u}$, and $g^{y_k} = g^{v_k} = (1; 0, 0, 1)$ for all $y_k \in \bar{y}$, $v_k \in \bar{v}$). We define the goal formula φ to be:

$$\varphi = \psi(\bar{x}, \bar{y}) \vee (\chi(\bar{u}, \bar{v}) \wedge w).$$

Now, I is a "yes"-instance iff 2 is a swing player for coalition $\{1\}$ in the BWVG we constructed. □

Dummy Players. We now move on from studying the impact of a player with respect to a *given* coalition to analysing his contribution to *all* coalitions in the game. A useful concept here is that of a *dummy* player: given a game G, a player i is said to be a *dummy* in G if $v(C) = v(C \cup \{i\})$ for all $C \subseteq N \setminus \{i\}$. In other words, i is a dummy if she is not a swing player for *any* coalition. Formally, an instance of DUMMY is a pair $\langle G, i\rangle$, where G is a BWVG and i is a player in G. It is a "yes"-instance if i is a dummy in G and a "no"-instance otherwise. When the game in question is restricted to be a WVG, the complexity of DUMMY is well understood: this problem is coNP-complete for weights given in binary and polynomially solvable for weights given in unary. In contrast to WVGs, for general BWVGs DUMMY is coNP-hard even for small weights. The result holds even if the underlying formula is a conjunction, i.e., for VWVGs. However, as the results in Section 4 will imply, the size of the formula must be nonconstant.

THEOREM 7. *The problem* DUMMY *is coNP-hard even if all weights in all component games are in $\{0, 1\}$, and G is an m-vector weighted voting game.*

PROOF. The reduction is from the classic NP-complete problem X3C (Exact Cover by 3-Sets) [8]. An instance of X3C is given by a ground set $\mathcal{E} = \{e_1, \dots, e_{3K}\}$, and a system of subsets $\mathcal{F} = \{S_1, \dots, S_\ell\}$, where $|S_i| = 3$, $S_i \subseteq \mathcal{E}$ for $i = 1, \dots, \ell$. It is a "yes"-instance if $\mathcal{E}$ can be covered by K sets from $\mathcal{F}$, i.e. there exists a set system $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}| = K$, such that for any $e \in \mathcal{E}$ there exists an $S_i \in \mathcal{F}'$ such that $e \in S_i$, and a "no"-instance otherwise. Given an instance of X3C $(\mathcal{E}, \mathcal{F})$, we construct a BWVG G with the set of players $N = \{1, \dots, \ell, \ell+1\}$, $3K+1$ component games $g^1, \dots, g^{3K+1}$, and $\varphi = p^1 \wedge \dots \wedge p^{3K+1}$. In the jth game, $j = 1, \dots, 3K$, we set $w_i^j = 1$ if $e_j \in S_i$ and $w_i^j = 0$ otherwise, and $T^j = 1$. In the last game g^{3K+1}, we set $w_i^{3K+1} = 1$ for $i = 1, \dots, \ell+1$ and $T^{3K+1} = K+1$. Finally, we set the player q to be tested for dummy status to be $q = \ell+1$. Observe that to win the jth game, $j = 1, \dots, 3K$, a coalition must contain a player i such that the corresponding set S_i covers e_j. Consequently, a coalition can win the first $3K$ games if and only if it corresponds to a valid cover of $\mathcal{E}$.

Suppose that we are given a "no"-instance of X3C. Then any set system that covers $\mathcal{E}$ contains at least $K+1$ sets. Now, consider any winning coalition C that includes player $\ell+1$. To win in the first $3K$ games, it has to correspond to a cover of $\mathcal{E}$, so $|C \setminus \{\ell+1\}| \geq K+1$. Hence, $C \setminus \{\ell+1\}$ is a winning coalition in the last game, too. As this is true for any C such that $\ell+1 \in C$, it follows that $\ell+1$ is a dummy. Conversely, if $(\mathcal{E}, \mathcal{F})$ is a "yes"-instance of X3C, consider a coalition C that corresponds to a cover of size K, and set $C' = C \cup \{\ell+1\}$. Clearly, C' is a winning coalition: the members of C ensure that the first $3K$ games are won, and player $\ell+1$ is needed to win the last game. On the other hand, C is too small to win the last game, so $\ell+1$ is a swing player for C and hence not a dummy. We conclude that a "yes"-instance of X3C corresponds to a "no"-instance of our problem and vice versa, and thus our problem is coNP-hard. □

Observe that for VWVGs (and, in fact, for BWVGs whose underlying formulas are monotone) this hardness result is tight: deciding whether a player is a dummy is in coNP. Indeed, to show that a player i is not a dummy, it suffices to guess a coalition C and check that (*i*) setting the variables controlled by C to $\top$ and all other variables to $\bot$ does not make φ true, but (*ii*) setting the variables controlled by $C \cup \{i\}$ to $\top$ and all other variables to $\bot$ does make φ true. By a similar argument, this problem is in coNP when the size of the formula can be assumed to be constant. On the other hand, for BWVGs with arbitrary formulas, the problem becomes much harder: one can modify the proof of Theorem 6 to show that checking whether a player is not a dummy is D_2^p-hard. In fact, checking if a player i is not a dummy appears to be harder than checking if she is a swing player for a given coalition, as we first have to guess a coalition, and then decide if i is a swing player for that coalition.

4. THE SHAPLEY VALUE

The *Shapley value* is an important solution concept for coalitional games [15], which in the context of weighted voting games provides a measure of voting power called the *Shapley–Shubik power index*. Intuitively, a player's Shapley value measures his marginal contribution to a randomly selected coalition, where the underlying probability model assigns equal probability to all orders in which players join the coalition.

Fix a game $G = (N, v)$, $|N| = n$, and let Π_n be the set of all possible permutations (orderings) of n agents. Each $\pi \in \Pi_n$ is a one-to-one mapping from $\{1, \dots, n\}$ to $\{1, \dots, n\}$. Denote by $S_\pi(i)$ the predecessors of agent i in π, i.e., $S_\pi(i) = \{j \mid \pi(j) < \pi(i)\}$. Shapley value of the ith agent in a game $G = (N, v)$ is denoted by sh_i^G (we omit the superscript G when the underlying game is clear from the context) and is defined as

$$\mathrm{sh}_i^G = \frac{1}{n!} \sum_{\pi \in \Pi_n} [v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))]. \qquad (3)$$

For simple games, sh_i^G counts the number of permutations π such that i is a swing player for the coalition $S_\pi(i)$.

Another popular measure of power is the *Banzhaf power index*. We omit the formal definition here due to space constraints, but remark that all our subsequent results on Shapley value also apply to the Banzhaf power index. Observe that the Shapley value of a player, as well as his Banzhaf power index, is 0 if and only if he is a dummy.

To use the Shapley value, one needs to understand the complexity of computing it. In WVGs, this problem is known to be hard (#P-complete) as long as the weights are given in binary [3]. This immediately implies that this problem is at least as hard in our setting. However, there is a polynomial-time algorithm for computing the Shapley value in WVGs with unary-encoded weights (see, e.g., [13]). It is thus interesting to ask whether this is also true of BWVGs. It turns out that this is unlikely. Indeed, we can use the argument from the proof of Theorem 7 to obtain the following:

COROLLARY 8. *Computing a player's Shapley value in a BWVG is* $\#\mathrm{P}$*-hard even if the game in question is a VWVG and all weights in all component games are in* $\{0, 1\}$.

PROOF. Observe that in the construction used in the proof of Theorem 7, q is a swing player for exactly N_K coalitions, where N_K is the number of exact covers of $\mathcal{E}$, and the size of each such coalition is exactly K. Hence, the Shapley value of player q is exactly $N_K \frac{K!(\ell+1-K)!}{(\ell+1)!}$, i.e., one can easily compute N_K given sh_q^G, ℓ, and K. As computing N_K is $\#\mathrm{P}$-complete [10], the statement follows. □

It seems that $\#\mathrm{P}$-hardness in the above corollary cannot be improved to $\#\mathrm{P}$-completeness, at least not in the case of unrestricted BWVGs. Intuitively, the reason for this is that computing Shapley value requires the ability to decide whether a given player is a swing for a given coalition, and this problem is D_2^p-complete for unrestricted BWVGs (Theorem 6). This intuition is captured more formally in the following theorem, which shows that computing the number of coalitions for which a player is a swing is not in $\#\mathrm{P}$ unless an unlikely complexity class collapse occurs. Typically, the complexity of computing this number is closely related to the complexity of computing the Shapley value (see, e.g., [7]) so we interpret the following theorem as suggesting that computing the Shapley value in unrestricted BWVGs is not in $\#\mathrm{P}$.

THEOREM 9. *Let* $G = \langle N, \mathcal{G}, \Phi, \varphi \rangle$ *be a BWVG and let* i *be a player in* N*. By* $f(G, i)$ *we mean the number of coalitions in* G *for which* i *is a swing player. If* f *is in* $\#\mathrm{P}$ *then* $\mathrm{NP} = \mathrm{UP}$.

PROOF. The idea of our proof is to show that if f is in $\#\mathrm{P}$ then a function h that given an X3C instance x returns 1 if x is a "yes"-instance and returns 0 otherwise is in $\#\mathrm{P}$. Since X3C is an NP-complete problem, existence of such a $\#\mathrm{P}$ function immediately implies $\mathrm{NP} = \mathrm{UP}$.

We give a reduction from h to f. Let x be a fixed input to f, i.e., let x be a fixed instance of X3C. Building on the techniques of Deng and Papadimitriou [3] and of Faliszewski and Hemaspaandra [7], it is easy to compute in polynomial time two nonnegative integers, k and n where $k < n$, and a WVG g with a player set $N = \{1, \dots, n, n+1\}$, weights $(w_1, \dots, w_n, 1)$ and threshold T such that (a) each coalition C such that $|C \setminus \{n+1\}| > k$ is winning, (b) each coalition C such that $|C \setminus \{n+1\}| < k$ is losing, (c) each coalition C such that $n+1$ is a swing for C contains exactly k players, and (d) player $n+1$ is a swing player for some coalition C if and only if x is a "yes"-instance of X3C and the set of coalitions for which $n+1$ is a swing player is in an easily-computable one-to-one correspondence with the set of solutions for x. We say that a binary string $b_1 b_2 \dots b_n$ spells a solution y for instance x if it holds that $b_1 b_2 \dots b_n = \chi_C(1)\chi_C(2) \dots \chi(n)$, where C is the coalition to which y uniquely corresponds and χ_C is the characteristic function of C. We say that a solution y for instance x is lexicographically maximal if the string that spells y is lexicographically maximal among all strings that spell solutions for x.

For each i in $\{1, \dots, n\}$, we set g^i to be a WVG that is won exactly by coalitions that include player i. For each i in $\{1, \dots, n\}$ we define h^i to be a WVG such that the only coalitions that win h^i are $\{1, \dots, n\}$ and the grand coalition $N = \{1, \dots, n, n+1\}$. It is easy to see that there are $g^1, h^1, \dots, g^n, h^n$ that satisfy the above description and that can be computed in time $\mathrm{poly}(n)$.

We form a BWVG $G = \langle N, \mathcal{G}, \Phi, \varphi \rangle$ where $\mathcal{G} = \{g, g^1, \dots, g^n, h^1, \dots, h^n\}$, $\Phi = \{u, u_1, \dots, u_n, v_1, \dots, v_n\}$, (variable u is associated with game g, variables $u_1, \ldots, u_n$ are associated with games $g^1, \ldots, g^n$, and variables $v_1, \ldots, v_n$ are associated with games $h^1, \ldots, h^n$), and $\varphi = u \wedge F(u_1, \ldots, u_n) \wedge F'(u_1, \ldots, u_n, v_1, \ldots, v_n)$. Predicates F and F' are defined as follows.

1. $F(u_1, \ldots, u_n)$ is true if and only if at least k of $u_1, \ldots, u_n$ are true.
2. $F'(u_1, \ldots, u_n, v_1, \ldots, v_n)$ is false if and only if $v_1 v_2 \ldots v_n$ spells a solution for x and $v_1 v_2 \ldots v_n$ lexicographically precedes $u_1 u_2 \ldots u_n$.

Note that it is not completely trivial (if at all possible, in the case of F') to express predicates F and F' as propositional formulas of polynomial size. We first describe our proof treating both F and F' as valid formulas and then show how to replace them with essentially equivalent formulas.

We claim that if x is a "yes"-instance then $f(G, n+1) = 1$ and otherwise $f(G, n+1) = 0$. First, by definition of G, it is easy to see that if x is a "no"-instance then for each coalition C it holds that C and $C \cup \{n+1\}$ control exactly the same variables and so $n+1$ is not a swing player for any coalition. Similarly, if x is a "yes"-instance then for each coalition $C \subseteq \{1, \ldots, n\}$ such that $|C| \neq k$ it holds that C and $C \cup \{n+1\}$ control the same variables. Indeed, the only variable that could potentially be controlled by $C \cup \{n+1\}$, but not by C, is u. On the other hand, for each coalition $C \subseteq \{1, \ldots, n\}$ it holds that if $|C| > k$ then C already controls u, whereas if $|C| < k$ then neither C nor $C \cup \{n+1\}$ controls u.

Thus, assume that x is a "yes"-instance and consider a coalition $C \subseteq \{1, \ldots, n\}$ such that $|C| = k$. Additionally, let us assume that $n+1$ is a swing player for game g (otherwise it is, again, easy to see that $n+1$ is not a swing player for C in G) and so (a) C is a losing coalition in G as C does not control variable u, and (b) $\chi_C(1)\chi_C(2)\ldots\chi_C(n)$ spells a solution for x.

We now claim that $C \cup \{n+1\}$ is winning if and only if $\chi_C(1)\chi_C(2)\ldots\chi_C(n)$ spells a lexicographically maximum solution for x; we omit the proof of this claim due to space restrictions. Together with the preceding discussion this establishes that if x is a "yes"-instance then $f(G, n+1) = 1$ and otherwise $f(G, n+1) = 0$. Since G can be computed in polynomial time, the assumption that f is in #P implies that $h(x)$ is in #P because to compute $h(x)$ it suffices to compute the game G and run the #P computation for $f(G, n+1)$.

To complete the proof it remains to show that we can in fact express predicates F and F' in G. The idea is that, via the standard proof that SAT is NP-complete (Cook's Theorem), for each polynomial-time computable predicate $P(y_1, \ldots, y_m)$, where $y_1, \ldots, y_m$ are Boolean variables, there is a polynomial-time computable Boolean formula $Q(x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that for each truth assignment to $y_1, \ldots, y_m$, $Q(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is satisfiable if and only if $P(y_1, \ldots, y_m)$ is true. Using this approach we can replace predicates F and F' with formulas H and H' that, given the same input as the respective predicates, are satisfiable if and only if the respective predicates hold. From inspection of the proof of Cook's Theorem's, it is easy to see that the proof of the current theorem, using predicates F and F', can be adapted to using formulas H and H' in a straightforward way. The only nontrivial part is deciding which coalitions control the "existentially quantified variables" of H and H'. In essence, we can set that any coalition controls the existentially quantified variables of H and that only the grand coalition controls the existentially quantified variables of H'. The details are technical but straightforward. $\square$

On the other hand, we can still compute Shapley value in polynomial time if both the weights are given in unary and the number of component games is bounded by a constant.

THEOREM 10. *Given a BWVG $G = \langle N, \mathcal{G}, \Phi, \varphi\rangle$ and a player $p \in N$, one can compute Shapley value of a player p in time $O((n^2 + s)(4nW)^m)$, where $|\mathcal{G}| = m$, $|\varphi| = s$, and $W = \max_{i,j} |w_i^j|$.*

PROOF. Our proof is based on dynamic programming. Without loss of generality, we can assume $p = n$. For any integer vector $\mathbf{z} = (z^1, \ldots, z^m) \in [-nW, nW]^m$, any $k = 1, \ldots, n-1$ and any $t = 1, \ldots, k$, let $N(\mathbf{z}, t, k)$ be the number of coalitions $C \subseteq N$ such that $w^j(C) = z^j$ for $j = 1, \ldots, m$, $|C| = t$, and $C \subseteq \{1, \ldots, k\}$. These values can be easily computed by induction on k. Indeed, for $k = 1$ we have $N(\mathbf{z}, t, 1) = 1$ if $t = 1$ and $w_1^j = z^j$ for $j = 1, \ldots, m$ and $N(\mathbf{z}, t, 1) = 0$ otherwise. Now, suppose that we have computed $N(\mathbf{z}, t, k)$ for all $\mathbf{z} \in [-nW, nW]^m$ and $t = 1, \ldots, k$. Set $\mathbf{z}_{k+1} = (z^1 - w_{k+1}^1, \ldots, z^m - w_{k+1}^m)$. We have

$$N(\mathbf{z}, t, k+1) = N(\mathbf{z}, t, k) + N(\mathbf{z}_{k+1}, t-1, k),$$

where the first summand counts coalitions that do not involve player $k+1$, and the second summand counts coalitions that do involve her. It is easy to see that we can compute all $N(\mathbf{z}, t, k)$ in time $O(n^2(2nW)^m)$.

Now, consider a coalition C, $|C| = t$, that satisfies $w^j(C) = z^j$ for $j = 1, \ldots, m$. By checking whether $z^j \geq T^j$ for $j = 1, \ldots, m$, we can identify the set Φ_C of variables controlled by C. By considering all possible truth assignments to variables in Φ_C and $\Phi \setminus \Phi_C$, we can decide whether C is a winning coalition. In a similar manner, we can decide whether $C \cup \{n\}$ is a winning coalition. We then set $I(\mathbf{z}, t) = v(C \cup \{n\}) - v(C)$; note that this value does not depend on the choice of C as long as $|C| = t$ and $w^j(C) = z^j$ for $j = 1, \ldots, m$. Computing the value of φ under a given truth assignment can be done in time $O(s)$, so for a fixed vector $\mathbf{z}$, $I(\mathbf{z}, t)$ can be computed in time $O(s2^m)$. Hence, all $I(\mathbf{z}, t)$, $\mathbf{z} \in [-nW, nW]^m$, $s = 1, \ldots, n-1$, can be computed in time $O(s2^m(2nW)^m)$.

Now, we can compute the Shapley value of player n as follows:

$$\mathrm{sh}_n = \frac{1}{n!} \sum_{\mathbf{z} \in [-nW, nW]^m} \sum_{t=1}^{n-1} N(\mathbf{z}, t, n-1) I(\mathbf{z}, t) t!(n-1-t)!$$

Indeed, $I(\mathbf{z}, t)t!(n-t-1)!$ counts the contribution to Shapley value by a coalition of size t that satisfies $w^j(C) = z^j$ for $j = 1, \ldots, m$, and $N(\mathbf{z}, t, n-1)$ counts the number of such coalitions. The overall running time of this procedure is $O(n^2(2nW)^m + s2^m(2nW)^m) = O((n^2 + s)(4nW)^m)$, as stated. $\square$

4.1 Combining Shapley Values of Component Games

Our consideration of the Shapley value in BWVGs thus far has ignored the fact that BWVGs have some structure, as defined by the goal formula. It is therefore interesting to ask whether, if we know a player's Shapley value in the component WVGs, we might be able to compute her Shapley value in the overall game by exploiting this structure somehow. In fact, we now show that Shapley value of a player in a BWVG may differ considerably from this player's Shapley values within component games. To simplify the exposition, in what follows, we consider games with nonnegative weights only; in such games, knowing the number of coalitions for which a given player is pivotal is sufficient for computing the values of his power indices.

First, let us introduce some useful notation. Fix a BWVG $G = \langle N, \mathcal{G}, \Phi, \varphi\rangle$, where $\mathcal{G} = \{g^1, \ldots, g^m\}$ is the set of component games, a player $i \in N$, and a component game g^k in $\mathcal{G}$. We write P_i^G to denote the set of permutations in which i is a swing player in G and we write $P_i^{G,k}$ to denote the set of permutations for which i is a swing player in g^k. Accordingly, by sh_i^G we mean the Shapley value of player i within game G, and by $\mathrm{sh}_i^{G,k}$ we mean his Shapley value in g^k. The goal of this section is to explore the relationship between sh_i^G and $(\mathrm{sh}_i^{G,1}, \dots, \mathrm{sh}_i^{G,m})$. We start with the following simple, yet very useful, observation.

THEOREM 11. *Let $G = \langle N, \mathcal{G}, \Phi, \varphi\rangle$ be a BWVG with $|\mathcal{G}| = m$, where φ is satisfiable but is not a tautology. For each $i \in N$ it holds that*

$$\bigcap_{k=1}^{m} P_i^{G,k} \subseteq P_i^G \subseteq \bigcup_{k=1}^{m} P_i^{G,k}.$$

PROOF. To prove this theorem it is enough to show that each permutation in $\bigcap_{k=1}^m P_i^{G,k}$ also belongs to P_i^G, and that no permutation outside of $\bigcup_{k=1}^m P_i^{G,k}$ is in P_i^G. Let us first handle the former issue.

Let $\pi \in \bigcap_{k=1}^m P_i^{G,k}$ be a permutation of players. This means that coalition $S_\pi(i)$ loses each game g^k, $1 \le k \le m$, but that coalition $S_\pi(i) \cup \{i\}$ wins each of them. That is, $S_\pi(i)$ does not control any of the propositions in Φ but $S_\pi(i) \cup \{i\}$ controls them all. Since φ is satisfiable but is not a tautology, this means that while $S_\pi(i)$ loses in G, $S_\pi(i) \cup \{i\}$ wins. That is, i is a swing player for permutation π in G. Thus, $\pi \in P_i$.

For the second part, let us assume that a permutation π of players from N does not belong to $\bigcup_{k=1}^m P_i^{G,k}$. This means that coalition $S_\pi(i) \cup \{i\}$ wins exactly the same component games as $S_\pi(i)$ and so $S_\pi(i) \cup \{i\}$ is winning in G if and only if $S_\pi(i)$ is as well. Thus, by definition, i is not a swing player for π in G. This completes the proof of the theorem. $\square$

Naturally, the above theorem applies to BWVGs whose underlying formulas are nonempty and monotone. So, in particular, it applies to VWVGs. Theorem 11 leads to the following tight bounds relating the Shapley values for BWVGs and their component games.

THEOREM 12. *Let $G = \langle N, \mathcal{G}, \Phi, \varphi\rangle$ be a BWVG where $\mathcal{G} = (g^1, \dots, g^m)$ and where φ is satisfiable but is not a tautology. For each player $i \in N$ it holds that $0 \le \mathrm{sh}_i^G \le \sum_{k=1}^m \mathrm{sh}_i^{G,k}$. These bounds are tight, even if φ is monotone (and, in fact, even if φ is either a conjunction or a disjunction of the propositions from Φ).*

PROOF. Fix a player i in N. Since all weights are assumed to be nonnegative, we have $0 \le \mathrm{sh}_i^G$. The inequalities $\mathrm{sh}_i^G \le \sum_{k=1}^m \mathrm{sh}_i^{G,k}$ follow directly from Theorem 11 Thus, it remains to show that the bounds are tight.

To this end, we provide four BWVGs, $G^{\wedge,n}$, $G^{\vee,n}$, $G^{\wedge,n+1}$ and $G^{\vee,n+1}$, that exemplify the interesting cases. These games are very similar; indeed, they differ only in the underlying formulas (conjunction or disjunction) and the thresholds of component games.

Let us fix a positive integer n, $n \ge 2$, and a set of players $N = \{1, \dots, n, n+1\}$. For each k, $1 \le k \le n$, and $T = n, n+1$, we define WVG g_T^k to be $(T; \underbrace{1, \dots, 1}_{k-1}, n, \underbrace{1, \dots 1}_{n-(k-1)})$ and set $G_T = \{g_T^1, \dots, g_T^n\}$. Finally, we set $\Phi = \{p^1, \dots, p^n\}$. Now $G^{\wedge,n}$, $G^{\vee,n}$, $G^{\wedge,n+1}$ and $G^{\vee,n+1}$ can be defined as follows:

1. $G^{\wedge,n} = \langle N, G_n, \Phi, \bigwedge_{k=1}^n p^i\rangle$,
2. $G^{\vee,n} = \langle N, G_n, \Phi, \bigvee_{k=1}^n p^i\rangle$.
3. $G^{\wedge,n+1} = \langle N, G_{n+1}, \Phi, \bigwedge_{k=1}^n p^i\rangle$,
4. $G^{\vee,n+1} = \langle N, G_{n+1}, \Phi, \bigvee_{k=1}^n p^i\rangle$.

We are interested in Shapley values for player $i = n+1$, i.e., the only player whose weight is 1 in each component game g_T^k.

We first consider game $G^{\wedge,n}$. To simplify notation, set $G = G^{\wedge,n}$ and let g^k be an arbitrary component game of $G^{\wedge,n}$. Let π be a permutation in $P_i^{G,k}$ and set $C = S_\pi(i)$. Since i's weight in g^k is 1, the threshold is n, and i is a swing player for C in g^k, it must be the case that C consists of the remaining $n-1$ players, i.e., $C = \{1, \dots, n\} \setminus \{k\}$. It is easy to see that C wins all the games $g^{k'}$, $k' \neq k$, $1 \le k' \le n$. This means that $\pi \in P_i^G$, and, moreover, the sets $P^{G,\ell}$, $1 \le \ell \le n$, are disjoint. Together with Theorem 11 this yields that $\mathrm{sh}_i^G = \sum_{\ell=1}^n \mathrm{sh}_i^{G,\ell}$.

On the other hand, the above analysis also shows that $\mathrm{sh}_i^{G^{\vee,n}} = 0$. Indeed, $G^{\vee,n}$ contains the same component games as $G^{\wedge,n}$, but in $G^{\vee,n}$ our goal function is a disjunction rather than a conjunction. Thus, a coalition wins $G^{\vee,n}$ even if it controls just a single variable. By Theorem 11, each permutation π such that i is a swing player for π in $G^{\vee,n}$ has to belong to $\bigcup_{\ell=1}^n P^{G^{\vee,n},\ell}$. However, by the above reasoning, if $\pi \in P_i^{G^{\vee,n},k}$ for some k, $1 \le k \le n$, then coalition $S_\pi(i)$ already wins all the component games except g_n^k (note that, naturally, $P_i^{G^{\vee,n},k} = P_i^{G^{\wedge,n},k}$). Thus, i is not a swing player for any permutation in $G^{\vee,n}$.

The reasoning for $G^{\wedge,n+1}$ and $G^{\vee,n+1}$ is similar. Let g_{n+1}^k be an arbitrary component game of $G^{\wedge,n+1}$ and let π be a permutation in $P_i^{G^{\wedge,n+1},k}$. Set $C = S_\pi(i)$. In g_{n+1}^k, there are n players, including i, with weight 1 and a single player, k, with weight n. Since the threshold in g_{n+1}^k is $n+1$, it is easy to see that $C = \{k\}$. However, both coalition C and coalition $C \cup \{i\}$ lose all other games $g_{n+1}^{k'}$, $k' \neq k$, $1 \le k' \le n$, as players i and k both have weight 1 in these games and the quota is $n+1$ (recall that $n \ge 2$). To win $G^{\wedge,n+1}$, a coalition needs to control all variables, i.e., win all component games, so neither C nor $C \cup \{i\}$ wins in $G^{\wedge,n+1}$. Thus, $\pi \notin P_i^{G^{\wedge,n+1}}$. Since π was chosen as an arbitrary member of $\bigcup_{\ell=1}^n P_i^{G^{\wedge,n+1},\ell}$ and by Theorem 11 $P_i^{G^{\wedge,n+1}} \subseteq \bigcup_{\ell=1}^n P_i^{G^{\wedge,n+1},\ell}$, it holds that $P_i^{G^{\wedge,n+1}} = \emptyset$ and so $\mathrm{sh}_i^{G^{\wedge,n+1}} = 0$.

On the other hand, it is easy to see that for each permutation π in $\bigcup_{\ell=1}^m P^{G^{\vee,n+1},\ell}$ it holds that $C = S_\pi(i)$ loses all component games of $G^{\vee,n+1}$, but $C \cup \{i\}$ wins exactly one of them. (This follows from the discussion above and the fact that component games in $G^{\wedge,n+1}$ and $G^{\vee,n+1}$ are identical.) A coalition wins $G^{\vee,n+1}$ if it controls at least one proposition in Φ, i.e., wins at least one component game, and so $P_i^{G^{\vee,n+1}} = \bigcup_{\ell=1}^m P^{G^{\vee,n+1},\ell}$. Since $P^{G^{\vee,n+1},1}, \dots, P^{G^{\vee,n+1},n}$ are all disjoint, this means that $\mathrm{sh}_i^{G^{\vee,n+1}} = \sum_{\ell=1}^n \mathrm{sh}_i^{G^{\vee,n+1},\ell}$. Shapley values of player i in games $G^{\wedge,n}$, $G^{\vee,n}$, $G^{\wedge,n+1}$ and $G^{\vee,n+1}$ jointly show tightness of our bounds. $\square$

Observe that the proof of Theorem 12 shows that a player can be a dummy in a BWVG even if he is not a dummy in either of the component games. In fact, his or her Shapley value can be smaller or greater than his Shapley values in each of the component games. This demonstrates that knowing a player's Shapley values in the component games does not immediately provide us with a way to compute his Shapley value in the overall game.

5. THE CORE

Another important consideration in cooperative game theory is that of the *stability* of a coalition. This is formally captured by the notion of *the core* [15, pp. 258–261], defined as follows. Given a game $G = (N, v)$, and a coalition $C \subseteq N$, we say that a vector $\mathbf{x} = (x_1, \ldots, x_{|C|}) \in \mathbb{R}^{|C|}$ is a C-feasible payoff vector if $\sum_{i \in C} x_i = v(C)$, where the values in $\mathbf{x}$ are assumed to be indexed by the elements of C. The *core* of a game G is the set of N-feasible payoff vectors $\mathbf{x}$ such that there is no coalition $C \subseteq N$ such that $v(C) > \sum_{i \in C} x_i$. The natural computational problems related to the core are checking whether a given payoff vector is in the core and verifying whether the core is nonempty.

Recall that games that can be represented by BWVGs are simple games, i.e., $v(C) \in \{0, 1\}$ for every $C \subseteq N$. Now, a well-known folk theorem tells us that a simple game has a nonempty core iff the game has a veto player: a player present in every winning coalition [15, p. 261]. Formally, a player i is a *veto* player if for any $C \subseteq N$, $v(C) = 1$ implies $i \in C$. Moreover, it is known that a payoff vector is in the core iff it distributes the value of the grand coalition N among the veto players. Hence, the decision problems mentioned above essentially reduce to checking whether a given player is a veto player. Now, if all weights are nonnegative, checking whether player i is a veto player is equivalent to checking whether $N \setminus \{i\}$ is a winning coalition. As argued earlier, this problem is easy for games where the goal formula φ is monotone or has constant size. However, if there are no restrictions on φ, our core-related problems become computationally hard.

Formally, in problem InCore we are given a BWVG G and a payoff vector $\mathbf{x}$ and we ask if $\mathbf{x}$ belongs to G's core, in CoreNonEmpty we are given a BWVG G and we ask if its core is nonempty, and in Veto we are given a BWVG G and a player i and we ask if i is a veto player in G.

Theorem 13. InCore, CoreNonEmpty *and* Veto *are* Π_2^p*-complete even if* $|N| = 2$ *and all weights in all component games are either* 0 *or* 1*. However, for non-negative weights these problems are in* P *if the underlying formulas are monotone.*

We omit the proof due to space restrictions. If we allow negative weights, the problem becomes hard even for vector weighted voting games: it is easy to see that checking whether there is a winning coalition in $N \setminus \{i\}$ in a 2-VWVG is at least as hard as Knapsack, and for VWVGs of nonconstant dimension this problem is NP-hard even for weights in $\{0, 1\}$ (reduction from 3-SAT). The problem can be solved in polynomial time for games of constant dimension with weights given in unary by dynamic programming.

6. CONCLUSIONS

We have introduced a model of decision making in multiple committees, in which it is assumed that each committee is a weighted voting game, and in which each committee is assumed to control a single issue. We have investigated a number of natural questions relating to this model. The model has some aspects in common with several other frameworks from the literature. *Boolean games* are simple games in which players have goals defined by Boolean formulas, and in which individual players control propositional variables [9]. The main differences are that in our framework, control is exercised by WVGs rather than individuals, and we also consider a single external goal. MC-nets, introduced by Ieong and Shoham [11], combine weights and logical formulas but in a different way than BWVGs. In MC-nets the variables correspond to agents and the weights correspond to formulas, whereas in BWVGs the variables correspond to games and the weights correspond to agents. The area of *judgment aggregation* considers the logical implications of decisions made by committees [12]. The main differences are that we consider decisions over multiple committees, and are concerned with the perspective of an external entity with a goal that they desire to see achieved. For future work, it might be interesting to combine elements of Boolean games with our framework (e.g., assuming that individual players have goals they desire to achieve). It would also be interesting to look at probabilistic extensions. Going in a different direction, we would like to compare, in some formal sense, the power of various formalisms used to represent games.

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